

On dispersion-corrected shallow water equations with varying bottom

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Highlights: Shallow water, varying bottom, improved dispersion, improved shoaling.

1 Introduction

We consider two-dimensional surface gravity waves in irrotational motion propagating at the surface of perfect incompressible fluid. The free surface at $y = \eta(x, t)$ and the bottom at $y = -d(x, t)$ are both impermeable, with x the horizontal variable, y the vertical upper one and t the time. Assuming long waves in shallow water (i.e., ∂_x and ∂_t are “small” operators) without restriction on their amplitudes (i.e., fully nonlinear), Serre [1] derived a set of approximate equations for constant depth. In presence of a varying bottom, these equations can be written [2]

$$h_t + [h\bar{u}]_x = 0, \quad (1.1)$$

$$\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x = \frac{1}{2}(\check{\gamma} + \check{\gamma})d_x - \frac{1}{3}h^{-1}[h^2\check{\gamma} + \frac{1}{2}h^2\check{\gamma}]_x, \quad (1.2)$$

where $h = \eta + d$ is the total water depth, \bar{u} is the depth-averaged horizontal velocity, $g > 0$ is the acceleration due to gravity, $\check{\gamma}$ and $\check{\gamma}$ being the vertical accelerations at, respectively, the bottom and the free surface, i.e.,

$$\check{\gamma} = -d_{tt} - 2\bar{u}d_{xt} - \bar{u}^2d_{xx} - (\bar{u}_t + \bar{u}\bar{u}_x)d_x, \quad (1.3)$$

$$\check{\gamma} = \check{\gamma} + h\{\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}\}. \quad (1.4)$$

The equation (1.1) for the mass conservation is exact, while the momentum equation (1.2) is an approximation: its left-hand side involves first-order terms and its right-hand side involves third-order terms. This approximation yields a fully nonlinear but only weakly dispersive model of water waves. It is thus desirable to improve the model for a better description of dispersive effects without increasing the mathematical complexity of the model, that is without introducing higher-order derivatives because they are computationally very demanding.

In order to improve the dispersive properties of shallow water models — i.e., to extend their validity to deeper water — some asymptotically consistent modifications of the momentum equations have been proposed. These modified equations involve free parameters that can be chosen to tune the (weakly dispersive) linear dispersion relation such that it better matches the (fully dispersive) exact relation (see, e.g., [3, 4, 5]). For the Serre equations, such modified equations can be obtained replacing the momentum equation (1.2) by [6, 7]

$$(1 - \alpha d^2 \partial_x^2)(\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x) = \frac{1}{2}(\check{\gamma} + \check{\gamma})d_x - \frac{1}{3}h^{-1}[h^2\check{\gamma} + \frac{1}{2}h^2\check{\gamma}]_x, \quad (1.5)$$

where α is a free parameter at our disposal. One can easily check that equation (1.5) is asymptotically consistent with (1.2). The parameter α is generally chosen considering a travelling wave of permanent form in constant depth. In such a case, the linear dispersion relation (relating the angular frequency ω and the wavenumber k) of the Serre equations is a (2, 2)-Padé approximation (in the wavenumber) of the exact relation, while when $\alpha \neq 0$ one gets a (4, 2)-Padé approximation

$$\frac{\omega^2}{g/d} = \frac{(kd)^2 + \alpha(kd)^4}{1 + (\frac{1}{3} + \alpha)(kd)^2}. \quad (1.6)$$

For all α , the linear dispersion relation matches the exact one $\omega^2 = gk \tanh(kd)$ at least up to the fourth-order in its Maclaurin expansion in terms of the wavenumber, but for $\alpha = 1/15$ the matching is up to the sixth-order. Thus, $\alpha = 1/15$ is the best choice to improve the dispersive properties (according to the criterion considered here).

However, this improvement occurs only for horizontal bottoms. Indeed, in presence of a bottom slope ($d_x \neq 0$), the dispersive properties of the modified equations are of fourth-order only for all α . This means that the modification (1.5) cannot improve the model in presence of a varying seabed, as one can check (see below). In practice the dispersive properties are nevertheless somewhat improved for very mild slopes ($|d_x| \ll 1$), but it is of practical interest to remove this restriction.

In this work, we propose another modification of the Serre momentum equation such that the dispersive effect are improved for finite constant slopes. To this aim, we first reduce, in section 2, the exact (i.e., fully dispersive) linear equations to a single pseudo-differential equation for the free surface only. In section 3, we consistently modify the Serre equations with an unknown operator that is determined by identification with the shallow water approximation of the exact linear equation. These modified Serre equations for varying depth should thus provide an improvement at least for mild-curvature of the seabed (i.e., when the gradient of the bottom slope is small).

2 Linear waves on sloping beach

The (fully dispersive) linearised Euler equations for an irrotational motion are

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for } -d \leq y \leq 0, \quad (2.1)$$

$$\phi_y + d_t + d_x \phi_x = 0 \quad \text{at } y = -d, \quad (2.2)$$

$$\phi_t + g\eta = 0 \quad \text{at } y = 0, \quad (2.3)$$

$$\phi_y - \eta_t = 0 \quad \text{at } y = 0, \quad (2.4)$$

where ϕ is a velocity potential. Here, we consider constant slopes, i.e., $d(x, t) = d_0 + sx$ where $d_0 \geq 0$ is the depth at $x = 0$ and $s \stackrel{\text{def}}{=} \tan(\theta_0)$ is the constant slope of the seabed ($\theta_0 \geq 0$ the seabed angle of inclination). Using the conformal mapping $z = x + iy \mapsto Z = X + iY$ where

$$Z \stackrel{\text{def}}{=} \frac{d_0}{\theta_0} \log\left(\frac{z}{d_0} + \frac{1}{s}\right), \quad (2.5)$$

the wedge domain $\{sx \geq -d_0; -d_0 - sx \leq y \leq 0\}$ is mapped onto the strip $-d_0 \leq Y \leq 0$ with $x = -d_0/s \mapsto X = -\infty$ and $x = +\infty \mapsto X = +\infty$. In the mapped variables, after elimination of $\eta(x, t) = -g^{-1}\phi_t(x, 0, t)$, the equations (2.1)–(2.4) become

$$\Phi_{XX} + \Phi_{YY} = 0 \quad \text{for } -d_0 \leq Y \leq 0, \quad (2.6)$$

$$\Phi_Y = 0 \quad \text{at } Y = -d_0, \quad (2.7)$$

$$\Phi_Y + g^{-1}\theta_0 \exp(\theta_0 X/d_0) \Phi_{tt} = 0 \quad \text{at } Y = 0, \quad (2.8)$$

where $\Phi(X, Y, t) \stackrel{\text{def}}{=} \phi(x(X, Y), y(X, Y), t)$.

The general solution of the Laplace equation (2.6) satisfying the lower boundary condition (2.7) is [8]

$$\Phi(X, Y, t) = \frac{1}{2} \check{\Phi}(Z + id_0, t) + \frac{1}{2} \check{\Phi}(Z^* - id_0, t) = \cos((Y + d_0) \partial_X) \check{\Phi}(X, t), \quad (2.9)$$

where $\check{\Phi}(X, t) \stackrel{\text{def}}{=} \Phi(X, -d_0, t)$ is the velocity potential at the bottom. Substituting (2.9) into the upper boundary condition (2.8), one gets the complex difference-differential equation

$$\check{\Phi}_X(X + id_0, t) - \check{\Phi}_X(X - id_0, t) + \frac{\theta_0}{ig} \exp\left(\frac{\theta_0 X}{d_0}\right) \left[\check{\Phi}_{tt}(X + id_0, t) + \check{\Phi}_{tt}(X - id_0, t) \right] = 0. \quad (2.10)$$

Exploiting the relation $\check{\Phi}(X \pm id_0, t) = \exp(\pm i d_0 \partial_X) \check{\Phi}(X, t)$ (Taylor expansion around $d_0 = 0$) and using the velocity potential at the free surface $\check{\Phi}(X, t) \stackrel{\text{def}}{=} \Phi(X, 0, t) = \cos(d_0 \partial_X) \check{\Phi}(X, t)$, this difference-differential equation can be rewritten as the real pseudo-differential equation

$$\tan(d_0 \partial_X) \check{\Phi}_X(X, t) - g^{-1} \theta_0 \exp(\theta_0 X/d_0) \check{\Phi}_{tt}(X, t) = 0. \quad (2.11)$$

Note that $A(X, t) \stackrel{\text{def}}{=} \eta(x(X, 0), t) = -g^{-1} \check{\Phi}_t(X, t)$ also satisfies the equation (2.11).

Returning to the physical variable $x(X, 0) = d_0 \exp(\theta_0 X/d_0) - d_0/s$, since $d = d_0 + sx$ and $s = \tan(\theta_0) = d_x$, the relation (2.11) is rewritten in terms of the original variables

$$\left\{ \frac{\partial^2}{\partial t^2} - g \frac{\partial}{\partial x} \tan\left(\frac{\arctan(d_x)}{d_x} d \frac{\partial}{\partial x}\right) \right\} \eta(x, t) = 0. \quad (2.12)$$

It should be emphasised that the equation (2.12) is exact for (fully dispersive) linear waves provided that $d_{xx} = d_t = 0$, i.e., for flat static bottoms, in particular for a constant depth. For more general seabeds, (2.12) provides a reasonable approximation if the bottom varies very slowly in time and if its curvature is small, such assumptions being made to derive shallow water approximations.

The relation (2.12) suggests to introduce an “apparent” (or “effective”) water depth D as

$$D \stackrel{\text{def}}{=} d_x^{-1} \arctan(d_x) d \leq d. \quad (2.13)$$

This shows that a bottom slope creates a slowdown of the wave compared to a flat bottom of the same depth, an effect conjectured by Dutykh and Clamond [9] from a non-dispersive shallow water model. In shallow water ∂_x is “small” and, since $d_{xx} = 0$, the operator $\tan(D \partial_x)$ can be expanded up to the fifth-order as

$$\begin{aligned} \tan(D \partial_x) &\stackrel{\text{def}}{=} D \partial_x + \frac{1}{3} D \partial_x D \partial_x D \partial_x + \frac{2}{15} D \partial_x D \partial_x D \partial_x D \partial_x D \partial_x + \dots \\ &\approx d \partial_x + \frac{1}{3} \partial_x d^3 \partial_x^2 + \frac{2}{15} d^5 \partial_x^5 + \frac{4}{3} d_x d^4 \partial_x^4 + 3 d_x^2 d^3 \partial_x^3 + d_x^3 d^2 \partial_x^2. \end{aligned} \quad (2.14)$$

This expansion should be compared with the approximate Serre-like equations in order to derive suitable improvements. We note in passing that some approximate and empirical relations used in engineering for water waves propagating over mild slopes could be somewhat improved replacing d by D , thus turning *mild-slope approximations* into *mild-curvature approximations*.

It should be noted that the equation (2.12) can be solved analytically for standing waves over constant slope [10], but this analytic solution is hardly tractable. Moreover, in order to improve shallow water approximations, special solutions are not needed. Indeed, it is sufficient, simpler and more general to compare the equations via their shallow water expansions, i.e., expansions such as (2.14) obtained assuming that ∂_x and ∂_t are “small” operators.

3 Modified Serre’s equations

In order to address the drawback of (1.5), a better modification is sought replacing $-\alpha d^2 \partial_x^2$ by another second-order differential operator \mathcal{D} to be defined such that the linearised equations match the (fully dispersive) exact ones for constant slopes up to the highest possible order in the shallowness expansion. Of course, this alternative modification of the momentum equation is also asymptotically consistent with the original equation, as one can easily check.

We thus consider infinitesimal waves with a fluid motion close to rest — i.e., η and \bar{u} are small — with $d = d(x)$ and $d_{xx} = 0$. The linearised modified Serre-like equations are then

$$\eta_t + [d \bar{u}]_x = 0, \quad (3.1)$$

$$(1 + \mathcal{D})(\bar{u}_t + g \eta_x) - \frac{1}{3} d^2 \bar{u}_{xxt} - d_x d \bar{u}_{xt} = 0, \quad (3.2)$$

and eliminating \bar{u} between these two relations, one gets after some algebra

$$\left\{ g^{-1} \partial_t^2 - \partial_x d \left[1 - \frac{1}{3} [1 + \mathcal{D}]^{-1} d^{-1} \partial_x d^3 \partial_x \right]^{-1} \partial_x \right\} \eta = 0. \quad (3.3)$$

The sixth-order shallow water expansion (i.e., assuming that ∂_x is “small”) of this linear equation

$$\left\{ g^{-1} \partial_t^2 - \partial_x d \partial_x - \frac{1}{3} \partial_x^2 d^3 \partial_x^2 + \frac{1}{3} \partial_x d \mathcal{D} d^{-1} \partial_x d^3 \partial_x^2 - \frac{1}{9} \partial_x^2 d^3 \partial_x d^{-1} \partial_x d^3 \partial_x^2 \right\} \eta \approx 0, \quad (3.4)$$

is to be compared with the exact (i.e., fully dispersive) linear relations (2.12)–(2.14) for constant slopes. Thus, the expansion (3.4) matches the exact one up to the sixth-order only if

$$\mathcal{D} = \frac{1}{5} d_x^2 - \frac{1}{15} d^{-1} \partial_x d^3 \partial_x. \quad (3.5)$$

The classical improvement is recovered on constant depth, as it should be. This choice is an improvement for constant slopes but, in practice, it should also improve the model when the bottom curvature is small (i.e., if $|dd_{xx}| \ll 1$) and, at least, when the bottom varies very slowly in time.

4 Discussion

Considering fully-dispersive linear waves on constant slopes, we propose a modification of the Serre equations such that their dispersive properties are improved. Here, we focus on Serre’s equations but similar modifications hold for any variant of shallow water (Boussinesq-like) approximations.

A priori, this improvement is not limited to mild slopes, as some previous works, but to mild curvatures of the seabed. The effectiveness of this approach will be investigated via numerical simulations. More specifically, we will focus on the shoaling over quite steep beaches where the present modified Serre equations should provide improvements. These numerical simulations, yet to be done, will be presented at the conference.

Further theoretical investigations can be performed. For instance, one can look for: (i) dispersion improvements for arbitrary (i.e., not only constant) slopes; (ii) a three-dimensional extension (i.e., two horizontal spacial dimensions); (iii) a variational derivation of the modified equations. These possible extensions will be discussed at the conference.

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